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A remark on the numerical solution of singular integral equations and the determination of stress-intensity factors

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SUMMARY

As is well-known, an efficient numerical technique for the solution of Cauchy-type singular integral equations along an open interval consists in approximating the integrals by using appropriate numerical integration rules and appropriately selected collocation points. Without any alterations in this technique, it is proposed that the estimation of the unknown function of the integral equation is further achieved by using the Hermite interpolation formula instead of the Lagrange interpolation formula. Alternatively, the unknown function can be estimated from the error term of the numerical integration rule used for Cauchy-type integrals. Both these techniques permit a significant increase in the accuracy of the numerical results obtained with an insignificant increase in the additional computations required and no change in the system of linear equations solved. Finally, the Gauss-Chebyshev method is considered in its original and modified form and applied to two crack problems in plane isotropic elasticity. The numerical results obtained illustrate the powerfulness of the method.

1. Preliminaries

The numerical solution of Cauchy-type singular integral equations appearing in several fields of mathematical physics has become a topic of intense research in recent years. A sufficiently extensive literature on the subject is contained in Ref. [1]. Here we will restrict our attention only to singular integral equations along an interval (a, b) of the real axis, which may be finite or infinite. In this case, the most frequently used technique for the numerical solution of a singular integral equation consists in approximating the integral terms by using an appropriate numerical integration rule and applying the singular integral equation at appropriately selected collocation points (see e.g. Refs. [2-6], as well as the references mentioned in Ref. [1]). After the determination of the unknown function at the abscissas used, this function is approximated along the whole integration interval through the Lagrange interpolation formula in its general form or after adaptation to the numerical integration rule used [5, 7-8]. Thus, if the number of abscissas used in numerical integrations is n, the unknown function of the singular integral equation is finally assumed to be a polynomial of degree $(n-1)$. Hence, if this function is a polynomial of degree greater than $(n-1)$, its approximation will not be exact.

On the other hand, generally Gaussian (Gauss-type, Radau-type and Lobatto-type) numerical integration rules are used for the numerical solution of singular integral equations, their accuracy being of the order of $2(n-1)$ and not only $(n-1)$ as happens with the Lagrange interpolation formula. In this way, although the values of the unknown function are sufficiently accurate at the abscissas used, where they are obtained directly from the numerical solution of the system of linear equations to which the singular integral equation is reduced, the accuracy of the unknown function at other points of the integration interval is not satisfactory because of the relatively low accuracy of the interpolation formula. Of course, in special cases, like crack problems in plane elasticity, it was suggested [4] that the points of the integration interval where the values of the unknown function are primarily required be included among the abscissas of the numerical integration rule used although this fact reduced slightly the accuracy in numerical integrations. In any case, the final numerical results at the points of interest were obtained with much more accuracy than that obtained when interpolation had to be used. For example, in the case of crack problems in plane isotropic elasticity, the application of the Lobatto-Chebyshev numerical integration rule, accurate for polynomial integrands of degree ($2n$-3), gives much more accurate results at the crack tips than the application of the Gauss-Chebyshev rule, accurate for polynomial integrands of degree ($2n$-1), but requiring the use of an extrapolation formula, accurate for polynomials of degree (n-1) only, for the estimation of the stress-intensity factors at the crack tips [4]. Furthermore, it is evident that in most cases the greater is the degree of a polynomial function for which a numerical technique is exact (like numerical integration or interpolation), the greater will be also the accuracy of the numerical results for non-polynomial functions since, in general, all well-behaving functions can be approximated by a polynomial.

Under these conditions, the authors feel that it is not reasonable to use highly accurate numerical integration rules in the numerical solution of singular integral equations and, hence, obtaining highly accurate values of the unknown function at the abscissas used, but having afterwards to approximate the unknown function along the whole integration interval by using the less accurate Lagrange interpolation formula. Two techniques to avoid this inconsistency, one based on the use of the Hermite interpolation formula and the other on taking into account the error term due to the Cauchy integral in numerical integration (as will become clear in the sequel) are proposed in this paper. It should be emphasized that these techniques concern only the interpolation scheme used at the points of the integration interval not coinciding with the abscissas and not the system of linear equations solved and the values of the unknown function at these abscissas. Since the formation and solution of this system of equations constitutes the main part in the numerical solution of a singular integral equation, it is evident that the proposed techniques improve the numerical results with almost no increase in the required number of numerical computations. (Of course, some computations are required even when applying the elementary Lagrange interpolation formula as was done up to now).

2. Application of the Hermite interpolation formula

We consider the following singular integral equation along the real interval (a, b):

$$B(x)w(x)g(x) + \int_a^b w(t)K(t,x)g(t)dt = f(x), \qquad a < x < b, \qquad (2.1)$$

where $B(x)$ and $f(x)$ are known functions, $w(x)$ is the weight function (determined in advance from theoretical considerations), $K(t, x)$ the kernel of the singular integral equation, which is assumed to consist of a Cauchy-type term of the form $A(x)/(t-x)$ and a regular Fredholm term $k(t, x)$, that is,

$$K(t,x) = A(x)/(t-x) + k(t,x). \tag{2.2}$$

By taking into account the results of Refs. [1-6] and particularly those of Ref. [6], we reduce Eq. (2.1) to the following system of linear equations:

$$\sum_{i=1}^{n} A_i K(t_i, x_k) y(t_i) = f(x_k), \quad k = 1, 2, \ldots, n_0, \tag{2.3}$$

where A_i and t_i are the weights and abscissas respectively of the numerical integration rule used (evidently compatible with the integration interval $[a, b]$ and the weight function $w(t)$):

$$\int_a^b w(t) g(t) dt = \sum_{i=1}^{n} A_i g(t_i) + E_n, \tag{2.4}$$

E_n denoting the error term, and $y(x)$ denotes the approximation of $g(x)$ along $[a, b]$. The number n_0 of Eqs. (2.3) may be equal to n or (n-1). In the latter case, one more linear equation resulting from some physical condition, e.g.:

$$\int_a^b w(t) g(t) dt = C, \tag{2.5}$$

where C is a constant, has to be taken into account. Finally, the collocation points x_k in Eqs. (2.3) are determined as the roots of the following generally transcendental equation:

$$q_n(x_k) = 0, \quad k = 1, 2, \ldots, n_0, \tag{2.6}$$

where

$$q_n(x) = \frac{1}{2} \int_a^b \frac{w(t) \sigma_n(t)}{x-t} dt \tag{2.7}$$

with

$$\sigma_n(x) = \prod_{i=1}^{n} (x - t_i) \tag{2.8}$$

if $B(x) \equiv 0$, and in a somewhat more complicated way if $B(x) \not\equiv 0$. These results have been well-established in Refs. [1-6] and the additional references mentioned in Ref. [6].

Here we wish to apply the Hermite interpolation formula [9, § 8.2] instead of the Lagrange interpolation formula. The Hermite interpolation formula, which serves also for the classical derivation of Gaussian numerical integration rules, has the form

$$y(x) = \sum_{i=1}^{n} h_i(x) y(t_i) + \sum_{i=1}^{n} h_i^*(x) y'(t_i), \tag{2.9}$$

where $y(t_i)$ and $y'(t_i)$ are the values of the function $y(x)$ under consideration and its derivative at the abscissas t_i used (which need not be related to Gaussian integration rules) and $h_i(x)$ and $h_i^*(x)$ denote the following functions [9, § 8.2]:

$$h_i(x) = [1-2\ell_i'(t_i)(x-t_i)]\ell_i^2(x), \tag{2.10}$$

$$h_i^*(x) = (x-t_i)\ell_i^2(x), \tag{2.11}$$

where

$$\ell_i(x) = \sigma_n(x)/[(x-t_i)\sigma_n'(t_i)], \tag{2.12}$$

$\sigma_n(x)$ being determined from Eq. (2.8). For comparison purposes, we can mention that the Lagrange interpolation formula, not taking into account the values of $y'(x)$ at the abscissas t_i, has the form [9, § 8.2]

$$y(x) = \sum_{i=1}^{n} \ell_i(x) y(t_i). \tag{2.13}$$

Of course, Eqs. (2.9) and (2.13) can be simplified when using special, and particularly Gaussian, numerical integration rules as was done for Eq. (2.13) in Refs. [5, 7-8].

The Hermite interpolation formula (2.9) is exact whenever $y(x)$ is a polynomial of degree $(2n\text{-}1)$, whereas the Lagrange interpolation formula (2.13) is exact whenever $y(x)$ is a polynomial of degree $(n-1)$ only as already stated. From the numerical solution of Eqs. (2.3), probably supplemented by the linear equation

$$\sum_{i=1}^{n} A_i y(t_i) = C, \tag{2.14}$$

resulting from Eq. (2.5), we can determine the approximations $y(t_i)$ to the values of $g(t_i)$. These values are exact only if $k(t, x)g(t)$ is a polynomial of degree m (with respect to t), where m is the maximum degree of a polynomial for which Eq. (2.4) is exact. If $k(t, x) \equiv 0$, then the values $y(t_i)$ are exact (that is $y(t_i) = g(t_i)$) if $g(x)$ is a polynomial of degree $(m + 1)$ as is clear from the developments of Refs. [4-6]. In any case, in most cases, and particularly in those when Gaussian integration rules are used, the numerical results $y(t_i)$ for $g(t_i)$ are sufficiently accurate to justify the use of the Hermite interpolation formula.

Now, in order to use this formula, (2.9), we have to determine the values $y'(t_i)$ approximating the derivative of the unknown function $g(t)$ at the abscissas used. This can be easily done by taking into account the numerical integration formula [10]

$$\int_a^b \frac{w(t)g(t)}{t-t_m} dt = \sum_{\substack{i=1 \\ i \neq m}}^{n} A_i \frac{g(t_i)}{t_i - t_m} + A_m g'(t_m) - 2g(t_m)\Lambda_n(t_m) + E_n, \tag{2.15}$$

which is valid for Cauchy-type principal value integrals only when the variable x coincides with an abscissa t_m $(m = 1, 2, \ldots, n)$. In this formula $\Lambda_n(x)$ is the function [10]

$$\Lambda_n(x) = [q_n'(x) + A_m \sigma_n''(x)/4]/\sigma_n'(x), \tag{2.16}$$

where $q_n(x)$ and $\sigma_n(x)$ are determined from Eqs. (2.7) and (2.8) respectively. More explicit formulas for $\Lambda_n(x)$ are given in Ref. [10] for some common numerical integration rules. By taking into account Eq. (2.15) and applying it to Eq. (2.1) (for $x = t_m$), we can find directly the values of $y'(t_m)$ at the abscissas t_m without solving any system of equations. Hence, we can now determine directly the interpolating function $y(x)$ from the Hermite interpolation formula (2.9) along the whole interval $[a, b]$. It is finally worth-mentioning that in the above procedure the abscissas t_m have been used as additional collocation points, besides the initial collocation points x_k ($k = 1, 2, ..., n_0$) determined from Eq. (2.6) (if $B(x) \equiv 0$) or some other similar equation (if $B(x) \not\equiv 0$).

3. Estimation of the unknown function from the error term

In some cases it is more convenient to determine the approximation $y(x)$ along $[a, b]$ without using interpolation formulas but directly from the part of the error term of the Cauchy-type integral due to the simple pole of the integrand inside $[a, b]$. By taking into account the following form of Eq. (2.4) valid for Cauchy-type principal-value integrals [10]:

$$\int_a^b \frac{w(t)g(t)}{t-x}\,dt = \sum_{i=1}^{n} A_i \frac{g(t_i)}{t_i-x} - 2g(x)K_n(x) + E_n, \quad x \neq t_i (i = 1, 2, \ldots, n), \tag{3.1}$$

where

$$K_n(x) = q_n(x)/\sigma_n(x), \tag{3.2}$$

and applying it to Eq. (2.1), we clearly see that, once the approximate values $y(t_i)$ of the unknown function $g(t)$ at the abscissas t_i have been determined from the numerical solution of the system of linear algebraic equations (2.4) (and, probably, (2.14)), then the application of Eqs. (2.4) and (3.1) to Eq. (2.1) for $x \neq t_i$ ($i = 1, 2, ..., n$), after $y(t_i)$ have been already determined, converts this equation into a formula providing $y(x)$ along the whole interval $[a, b]$. In this case, the value $f(x_0)$ of the right-hand side function at a point $x = x_0$ is evidently always taken into account for the estimation of $y(x_0)$. It should also be mentioned that this technique, not making use of interpolation formulas, is analogous to the corresponding technique used long ago for Fredholm integral equations of the second kind [11]. Finally, we can remark that this technique cannot be successfully used if the integral in Eq. (2.1) contains one or more simple poles near the integration interval. Such poles contribute significantly to the error term in numerical integrations. Of course, their influence can be taken into account, in an approximate way, as proposed in Ref. [12].

A third method for estimating $g(x)$ along $[a, b]$ can also be used. This method consists in using the Lagrange interpolation formula (2.13) based both on the values of $y(t_i)$ at the abscissas t_i (determined from the system of Eqs. (2.4) and, probably, (2.14)) and on the values of the same function at an arbitrary number of other points of the interval (a, b) determined as proposed in the previous paragraph. It is recommended that the number of these additional points is $(n\text{-}1)$, n or $(n + 1)$ and coincide with the middle-points of the subintervals in which the interval $[a, b]$ is divided by the abscissas used or other reasonably selected points. In this way,

$g(x)$ is now approximated by an interpolation polynomial $y(x)$ of degree $(2n\text{-}2)$, $(2n\text{-}1)$ or $2n$, completely compatible with the accuracy of Gaussian numerical integration rules.

4. Application to crack problems

As an application we consider crack problems in the classical isotropic plane elasticity. These problems are easily reducible to Cauchy-type singular integral equations (see e.g. [3-5]). The values of the unknown function of the singular integral equation at the end-points of the integration interval (that is at the crack tips) are proportional to the values of the corresponding stress-intensity factors k at these tips, which are of particular importance in fracture mechanics. At first, the Gauss-Chebyshev numerical integration rule has been widely used (and remains in use) for the solution of the singular integral equations of crack problems and the determination of the stress-intensity factors at the crack tips [2, 3]. In Ref. [4] it was proposed that this rule could be replaced by the Lobatto-Chebyshev rule for a more accurate determination of the stress-intensity factors. In fact, this rule was of the form (2.4) with $[a, b] \equiv [-1, 1]$ and

$$w(t) = (1-t^2)^{-\frac{1}{2}}, \tag{4.1}$$

$$t_i = \cos[(i-1)\pi/(n-1)], \quad i = 1, 2, \ldots, n, \tag{4.2}$$

$$A_i = \pi/(n-1), \quad i = 2, 3, \ldots, n-1, \quad A_1 = A_n = \pi/[2(n-1)], \tag{4.3}$$

whereas the collocation points x_k were determined from Eq. (2.6) as

$$x_k = \cos[(k-0.5)\pi/(n-1)], \quad k = 1, 2, \ldots, n-1, \tag{4.4}$$

under the valid assumption that Eq. (2.1) was of the first kind. As clear from Eq. (4.1), the end-points ($\pm$ 1) of the integration interval $[-1, 1]$ (representing the crack tips) were included among the abscissas used and, thus, the stress-intensity factors were determined in general with higher accuracy, the remarks of Sec. 1 taken also into account.

Yet, the Gauss-Chebyshev numerical integration rule is accurate for polynomial integrands of degree $(2n\text{-}1)$ for regular integrals and degree $2n$ for Cauchy-type principal value integrals [4], while the corresponding numbers for the Lobatto-Chebyshev numerical integration rule are $(2n\text{-}3)$ and $(2n\text{-}2)$. This means that, if we avail ourselves of the results of this paper, we can obtain more accurate values for the stress-intensity factors by using the technique based on the Gauss-Chebyshev rule (properly modified) than by using the technique based on the Lobatto-Chebyshev rule in a series of cases, contrary to what would be expected in the past. Two simple examples will illustrate this fact.

We consider the simple problem of a straight crack of length 2α inside a plane isotropic elastic medium. This problem is described by the Cauchy-type singular integral equation (see e.g. [3,5])

$$\frac{1}{\pi}\int_{-1}^{1} \frac{w(t)g(t)}{t-x}\, dt = -f(x), \tag{4.5}$$

where $f(x)$ is the loading distribution along the crack edges and $w(t)$ is given by Eq. (4.1). This singular integral equation was solved both by using the Gauss-Chebyshev method [2] with

$$t_i = \cos[(i-0.5)\pi/n], \quad i = 1,2,\ldots,n, \tag{4.6}$$

$$A_i = \pi/n, \quad i = 1,2,\ldots,n, \tag{4.7}$$

$$x_k = \cos(k\pi/n), \quad k = 1,2,\ldots,n-1, \tag{4.8}$$

and by using the Lobatto-Chebyshev method described by Eqs. (4.1-4.4). Of course, the condition of single-valuedness of displacements,

$$\int_{-1}^{1} w(t)g(t)dt = 0, \tag{4.9}$$

has been taken also into account.

The reduced values of the stress-intensity factors $k_I/(\sigma\alpha^{\frac{1}{2}}) = \pm g(\pm 1)$, where σ is a constant and the sign (+) is valid for $x = +1$ whereas the sign (−) is valid for $x = -1$ ([5]), were obtained for $f(x) = \sigma x^4$, as well as for $f(x) = \sigma \exp x$. These results are presented in the second and third columns of Tables 1 and 2 respectively with $n = 3(1)6$ in the first case and $n = 3(1)10$ in the second case. Of course, in the case of application of the Gauss-Chebyshev method the stress-intensity factors were determined at the crack tips by the simple extrapolation method reported in Ref. [8]. Finally, in the fourth columns of Tables 1 and 2 the values of the stress-intensity factors were determined *on the basis of the same values of* $y(t_i)$ as in the first column of these tables (that is by using also the Gauss-Chebyshev method) but by applying the technique presented in the first paragraph of Sec. 3. Thus, from Eqs. (3.1), (4.5) and (4.7) we obtain

$$g(\pm 1) \simeq y(\pm 1) = \mp \left[f(\pm 1) + \frac{1}{n} \sum_{i=1}^{n} \frac{y(t_i)}{t_i \mp 1} \right] / n, \tag{4.10}$$

TABLE 1

Reduced values of the stress-intensity factor at the tips $x = \pm 1$ of a simple crack under normal loading $f(x) = \sigma x^4$.

Method	Gauss-Chebyshev	Lobatto-Chebyshev	Modified Gauss-Chebyshev
n		$k_I/(\sigma\alpha^{1/2}) = \pm y(\pm 1)$	
3	0.062500	0.250000	0.375000
4	0.250000	0.375000	0.375000
5	0.312500	0.375000	0.375000
6	0.375000	0.375000	0.375000
Theoretical value		0.375000	

TABLE 2

Reduced values of the stress-intensity factors at the tips $x = \pm 1$ of a simple crack under normal loading $f(x) = \sigma \exp x$.

Method	Gauss-Chebyshev	Lobatto-Chebyshev	Modified Gauss-Chebyshev
n		$k_I/(\sigma\alpha^{1/2}) = \pm y(\pm 1)$	
		Crack tip $x = +1$	
3	1.648721	1.803313	1.831543
4	1.803313	1.830907	1.831227
5	1.827922	1.831223	1.831225
6	1.830907	1.831225	1.831225
7	1.831199	1.831225	1.831225
8	1.831223	1.831225	1.831225
9	1.831225	1.831225	1.831225
10	1.831225	1.831225	1.831225
		Crack tip $x = -1$	
3	0.606531	0.717871	0.700679
4	0.717871	0.701135	0.700905
5	0.698690	0.700908	0.700907
6	0.701135	0.700907	0.700907
7	0.700887	0.700907	0.700907
8	0.700908	0.700907	0.700907
9	0.700907	0.700907	0.700907
10	0.700907	0.700907	0.700907

where $y(x)$ denotes again the approximation of the unknown function $g(x)$ obtained from the numerical solution of Eqs. (4.5) and (4.9). To obtain Eq. (4.10), we have taken into account that in the case of the Gauss-Chebyshev method we have [10]

$$q_n(x) = -(\pi/2)U_{n-1}(x), \tag{4.11}$$

$$\sigma_n(x) = T_n(x), \tag{4.12}$$

where $T_n(x)$ and $U_n(x)$ denote the Chebyshev polynomials of the first and the second kind respectively. Hence

$$K_n(x) = -(\pi/2)U_{n-1}(x)/T_n(x) \tag{4.13}$$

and further

$$K_n(\pm 1) = \mp n\pi/2. \tag{4.14}$$

The superiority of the results of this modified form of the Gauss-Chebyshev method over the results of the Gauss-Chebyshev method in its ordinary form, as well as the results of the

Lobatto-Chebyshev method is clearly seen from Tables 1 and 2. In Table 1 the unknown function $g(x)$ is evidently a polynomial of fifth degree and the exact value for the stress-intensity factors $k_I/(\sigma\alpha^{\frac{1}{2}}) = 0.375$ is obtained by the three methods used in accordance with their accuracies, $(n\text{-}1)$, $(2n\text{-}2)$ and $2n$ respectively. Similarly, in Table 2 the unknown function $g(x)$ is evidently a polynomial of infinite degree, like the exponential function, but the relative accuracies of the three methods used remain unchanged. The modified Gauss-Chebyshev method is slightly superior to the Lobatto-Chebyshev method and of about double accuracy compared with the Gauss-Chebyshev method in its ordinary form.

5. Two remarks

The authors have unsuccessfully tried to generalize their present results to the case of Cauchy-type singular integrodifferential equations considered in Ref. [13]. As is clear from the results of this reference, if a numerical integration rule with n abscissas is used, the unknown function is determined at $2n$ points of the integration interval and not only n as happens in the case of Cauchy-type singular integral equations. Hence, the interpolating function $y(x)$ is a polynomial of degree $(2n\text{-}1)$ and any further attempt to increase this degree by using techniques similar to those of the present paper has no essential meaning.

Finally, it is evident that the results of the present paper are also applicable to the case of systems of uncoupled Cauchy-type singular integral equations. Such systems of equations arise often in practical problems.

REFERENCES

[1] N. I. Ioakimidis and P. S. Theocaris, The numerical evaluation of a class of generalized stress intensity factors by use of the Lobatto-Jacobi numerical integration rule, *International Journal of Fracture* 14 (1978) 469-484.

[2] F. Erdogan and G. D. Gupta, On the numerical solution of singular integral equations, *Quarterly of Applied Mathematics* 29 (1972) 525-534.

[3] F. Erdogan, G. D. Gupta and T. S. Cook, Numerical solution of singular integral equations; contained in: G. C. Sih (ed.), *Methods of analysis and solutions of crack problems* (Vol. 1 in the series *Mechanics of fracture*), Noordhoff, Leyden (1973), Chap. 7, pp. 368-425.

[4] P. S. Theocaris and N. I. Ioakimidis, Numerical integration methods for the solution of singular integral equations, *Quarterly of Applied Mathematics* 35 (1977) 173-183.

[5] N. I. Ioakimidis, *General methods for the solution of crack problems in the theory of plane elasticity*, Doctoral thesis at the National Technical University of Athens, Athens (1976) [Univ. Micr. order no. 76-21,056].

[6] P. S. Theocaris and N. I. Ioakimidis, Numerical solution of Cauchy-type singular integral equations, *Transactions of the Academy of Athens* 40 (No. 1) (1977) 1-39.

[7] D. F. Paget and D. Elliott, An algorithm for the numerical evaluation of certain Cauchy principal-value integrals, *Numerische Mathematik* 19 (1972) 373-385.

[8] S. Krenk, On the use of the interpolation polynomial for solutions of singular integral equations, *Quarterly of Applied Mathematics* 32 (1975) 479-484.

[9] F. B. Hildebrand, *Introduction to numerical analysis*, Tata McGraw-Hill Co. Ltd., New Delhi (1956), pp. 314-317.

[10] N. I. Ioakimidis and P. S. Theocaris, On the numerical evaluation of Cauchy principal value integrals, *Revue Roumaine des Sciences Techniques – Série de Mécanique Appliquée* 22 (1977) 803-818.

[11] L. V. Kantorovich and V. I. Krylov, *Approximate methods of higher analysis*, Interscience Publishers and P. Noordhoff, Groningen (1958), pp. 98-103.
[12] N. I. Ioakimidis and P. S. Theocaris, On the selection of collocation points for the numerical solution of singular integral equations with generalized kernels appearing in elasticity problems, *Computers and Structures* (to appear).
[13] N. I. Ioakimidis and P. S. Theocaris, On the numerical solution of singular integrodifferential equations, *Quarterly of Applied Mathematics* (to appear).